Student Understandings in Differential Calculus

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This paper considers the effectiveness of the traditional "first principles" approach to differential calculus and investigates some of the conceptual difficulties associated with it. Materials developed by Mary Barnes were adapted to produce a sequence of lessons characterised by investigative exercises and realistic calculus problems for a group of year 11 students. The lessons proved successful in improving students' conceptual understanding without impairing their ability to perform standard calculus techniques. They also reported positive feelings towards their study of calculus.

In recent years there has been a trend towards developing a more meaningful mathematics curriculum which is relevant to the needs and experiences of students. This has involved designing activities which are of interest to the students and which allow them to explore mathematical concepts for themselves as they play a more active role in their own learning. There has been a shift away from traditional teacher-centred methods in which the rote learning of rules and formula predominates, to one where the emphasis is on discovering patterns and understanding concepts. One area where such curriculum reform has been very clearly demonstrated is the study of calculus and a great deal of resources has been allocated to designing more effective means of teaching this topic.

There are important reasons why the calculus curriculum is in need of reform. The calculus component included in secondary schools has gradually been increased and made available to a wider cross-section of the student population (Grimison, 1990). As well, higher retention rates in secondary schools have meant that larger numbers of students are exposed to calculus instruction than has previously been the case (White, 1992) and for the majority of these students an abstract teaching of calculus concepts is inappropriate to their mathematical ability and educational needs.

Improvements in our understanding of the ways that students best learn new mathematical concepts have also called into question the effectiveness of the traditional approaches to calculus instruction. Many students still view mathematics as a body of rules having no relation to each other and they must be challenged to see the relevance of the mathematics they are studying as well as its connection to the mathematical concepts they are already familiar with (Mitchelmore & White, 1995).

The Traditional Approach

The traditional "first principles" approach to calculus instruction begins with a diagram like that shown in Figure 1 where the gradient of the tangent to the curve at the point P is the limiting value of the gradient of the secant drawn through P as Q approaches P. This is the method outlined in the New South Wales Department of Education syllabus for the 2/3 unit course (1982) and it is followed closely in the majority of high school texts which give a purely algebraic explanation using function notation and limits. There is generally no attempt to relate numerical and graphical explorations of the concepts and the main purpose seems to be to proceed immediately to the rules for differentiation (Wilkins, 1987).

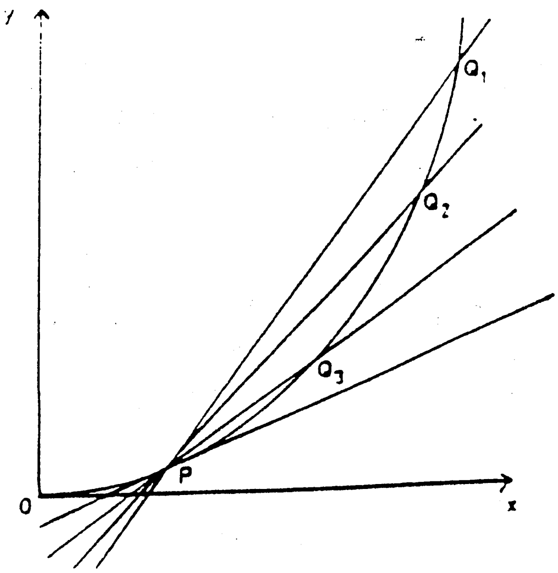


Figure 1.

Ryan (1991) reports that a large number of students who are instructed using a diagram like Figure 1 do not see the limiting process at all since they focus almost exclusively on the *chord* PQ and therefore do not see the *secant* approaching the tangent. In the minds of these students, the interval PQ is gradually getting shorter and shorter until it disappears altogether, and the central idea of the limiting process is lost.

Although the essential aim of the first principles approach is to develop a method for finding the gradient of the tangent to a curve, the usual manner of presenting this is almost completely abstract and given without any real justification as to why it might be useful or important. The applications treated are also generally mathematical in nature and no attempt is made to relate them to the experiences of the students or even the mathematical knowledge which they already have. Presenting new concepts in such a way, devoid of any real meaning for the students, can only increase the likelihood of misunderstandings and make the learning process more complicated.

The traditional approach is further characterised by abstract and complex algebraic manipulations which expose students to some highly demanding mathematics such as functions and limits from the very beginning. The difficulties associated with these concepts and the complex notation used to present them obscure the calculus itself making it unlikely that students will gain any appreciation for its significance. There is an increased chance that the students will too quickly come to rely on learning the rules for differentiation by rote without any real understanding of their meaning or significance.

One of the hardest concepts the students encounter in the first principles process is that of a limit. The difficulty begins with the many informal notions of the word "limit" itself which the students already possess (Barnes, 1991). Many of these capture some aspect of the formal mathematical concept but may also distort the finer nuances of the definition creating a conceptual barrier which makes learning about limits more difficult.

Tall and Vinner (1981) describe the kind of cognitive conflict which can develop due to discrepancies between the *concept image* (preconceived ideas in the mind of the student) and the *concept definition* (the more precise mathematical idea). They found that students who held strongly entrenched concept images which conflicted in varying degrees with the concept definition were unable to resolve the incompatibility.

Trying to water-down the limit concept and use a more informal version of the definition does not help because of the danger of becoming too colloquial and imprecise (Tall & Schwarzenberger, 1978). The idea of a limit loses its precision and the conceptual difficulty associated with limits is actually increased. There is also the problem of the notation used in writing limits which can be problematic for some students. Even those who can successfully evaluate limits often show a serious lack of understanding of the geometric significance of what they are doing (Ferrini-Mundy & Graham, 1991). Yet despite all of these difficulties, a proper understanding of the limit concept is crucial to the first principles approach.

The first principles definition also incorporates a quite detailed and involved use of functions, however many students commence their study of calculus with a poorly developed understanding of functions and function notation. Students tend to view functions in a rather static way, regarding the algebraic and graphical representations of functions independently (Dreyfus & Eisenberg, 1982). They often regard a function as a formula or equation into which a single value should be substituted and lack any real appreciation of the function as having meaning for a number of values.

Eisenberg (1991) found that while most students could draw the graph of simple functions, they were unable to see the connection with its algebraic form and regarded the graph as something quite apart from the function itself. He described the power of visual images in developing the function concept and proposed that the lack of a strong visual image of a function can restrict the students to purely symbolic representations which may impede a full appreciation of the more dynamic nature of functions. The requirement of the first principles approach that students be able to recognise the secant lines moving along the curve as they approach the tangent is thus unlikely to be met.

One of the key building blocks for work involving functions and limits is the concept of a variable. White (1992) studied students' understanding of variables in a calculus setting and reported that their lack of success in solving rate of change problems often arose from a poor understanding of variables. When problems were posed in symbolic form, few students had difficulty in executing the correct procedures. However, when similar problems were presented without the algebraic framework, the number of incorrect responses increased significantly. His research suggests that a more graphically oriented approach to introductory calculus based on rates of change provides a better alternative than the more traditional algebraic approach.

Clearly the traditional first principles method of introducing students to the concepts of differential calculus is problematic and calculus teaching which continues to rely heavily on this approach will not provide an adequate basis from which to develop strong conceptual understanding. The combination of increased numbers of students who are inadequately prepared for a study of calculus and a method of instruction which incorporates too many difficult mathematical ideas right from the very beginning exacerbates the situation. As a result, the first principles approach too easily lends itself to a teacher-centred form of instruction where students rote learn the rules for differentiation without understanding them.

Methods of calculus instruction which take account of the conceptual difficulties outlined above and present the calculus material in a more realistic context which reflects the interests and experiences of the students are clearly required.

A New Approach

The materials used to produce the teaching program for this research were largely derived from the work of Mary Barnes, particularly her unit *Investigating Change* (1991). The essential elements of her approach are to avoid the conceptual difficulties associated with the first principles method and to give the calculus a real-world context through the use of meaningful examples. Discovery learning and group investigations are included to give the students a sense of the power of the calculus they are studying to solve realistic problems.

The sequence of lessons created as part of this research is further distinguished by some important ideas developed by David Tall (1991). The first is the notion that sufficiently magnifying a small section of certain curves shows them to be locally "straight". The limiting process is implicitly contained in the magnification of the curve but no direct mention of limits is required. Instead, the students use the zoom function on a graphics calculator to obtain a visual image of this local straightness and by examining the curve in this way, they can recognise the gradient of this "straight" magnified portion as an approximation to the gradient of the curve.

This local straightness of curves then allows us to distinguish between the *practical* tangent which is a line drawn through two points very close to each other on the curve, and the *theoretical* tangent which is the straight line through the point having the same gradient as the curve at that point. Students can obtain a good approximation to the gradient of the curve by calculating the gradient of the practical tangent numerically using the standard result from co-ordinate geometry and this approximation can be used to measure the instantaneous rate of change.

Method

Subjects

The subjects were a group of 24 secondary school students in an average ability Year 11 3-unit mathematics class at an independent girls' high school in metropolitan Sydney. None of the students had any previous experience in calculus. The class was taught by the researcher and the teaching program was conducted for 13 fifty minute lessons over 3 weeks. The students were aware of the experimental nature of the work they were doing.

Instruments

Two tests were designed for this study. The pre-test was administered immediately prior to the commencement of the teaching program and was a written test consisting of 4 questions which the students had twenty minutes to complete. The purpose of this test was to assess the students' understanding of some fundamental pre-calculus concepts. The post-test was administered twice, once at the conclusion of the teaching program (August) and again after a further period of ten weeks (October). The post-test was exactly the same on both occasions and did not include any of the questions from the pre-test. It was a written test consisting of 8 questions which the students had to complete in thirty minutes. The purpose of the post-test was to assess the students' competence with and understanding of the calculus concepts they had studied. The questions from the post-test are found in Table 1.

Data Collection

Information about the understandings which the students were gaining of the calculus concepts they studied was also gathered from other sources. Four students whose ability reflected the range of the class were selected to take part in two interviews. The students were interviewed individually for 10 minutes at the end of the first week of the teaching experiment and then again at its conclusion.

Another separate group of four students volunteered to keep a diary of their impressions and feelings for the calculus they were studying. These students were also of mixed ability. As well, the researcher kept his own diary in which observations of the classroom activities and the ways the students responded to them were recorded.

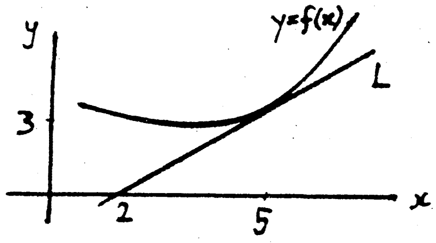
Table 1
Post-test questions

1. What is the gradient of the curve at the point (2,10)?

2. Find the equation of the tangent to the curve at the point where x=1.

3. Find the stationary points on and determine their nature. Sketch the curve.

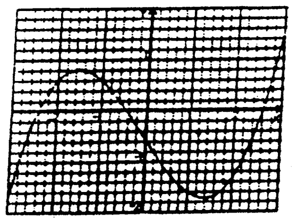
4. Suppose the line L is drawn as a tangent to the curve at the point (5,3) as shown in the diagram.
What is the value of
(a)
(b)



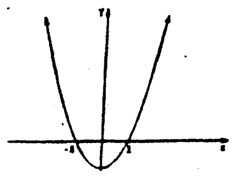
5. A fire has begun to spread through an area of bushland. Suppose that the area covered by the fire is given by where *t* is the number of days since the fire began. Explain how you would calculate the rate at which the area covered by the fire was increasing 10 days after it began.

6. Suppose that the function represents the number of people unemployed *t* months after the election of a new government. Translate each fact about the graph of into statements about the unemployment situation.
(a) The *y*-intercept of is 200 000
(b)
(c) The gradient of at is 1000

7. Here is a graph of a function .
From the graph, estimate the value of the derivative at .
Now, from looking at the graph of
sketch a graph of its derivative.



8. The diagram shows the graph of the derivative of a function . At what value(s) does have a minimum turning point?
Give a reason for your answer.



Results

The results of the pre-test suggested that the students generally had a good understanding of distance-time graphs and that they were able to match the gradient of these graphs to written information about rates of change. One of the questions on the pre-test directly imitated the first principles approach as demonstrated in Figure 1 in order to test whether the students were indeed able to see the limiting process occurring for themselves. The most common response was that the chord became progressively smaller until it disappeared which supports the findings of Ryan (1991) quoted earlier.

The results of the post-test can be considered in terms of the questions which related to standard differential calculus procedures and those which examined conceptual understanding. The former type were found in the first three questions of the post-test which required the students to (1) find a numerical value for the gradient of a curve whose equation was given (100% of students correct in both administrations of the test), (2) find the equation of the tangent to a curve at a given point (92% correct in August and 83% in October), and (3) locate and classify the stationary points on a curve in order to sketch its graph (79% correct in August and 88% in October). The results of these questions suggest that the students were proficient in performing these calculus techniques and that they successfully retained this proficiency over time.

The remaining five questions of the post-test considered the levels of understanding of the calculus concepts which the students had attained. Question 4 tested whether the students could interpret a geometric representation of the derivative given in function notation. In August, 29% of the students were correct and in October, 54% were correct. This indicates that the students had difficulty with function notation.

Questions 5 and 6 were designed to test the students' understanding of the relationship between the derivative, the gradient of a curve at a point and the rate of change. In both questions, 80% of the students gave a correct response indicating that the students had a very good grasp of these ideas.

Question 7 asked the students to sketch the graph of the derivative for a function whose graph was given. Although this is a conceptually demanding task, the students had practised it during the teaching program and this was reflected in their high success rate (75% in August and 88% in October). The increased percentage of correct responses suggests that their success was not merely because they had seen it recently in class.

The final question on the post-test, question 8, gave the students the graph of a derivative and asked them to interpret it. This was a difficult question which was of a type that the students had not previously seen (46% of students were successful in August and 58% in October). The phenomenon of an increased number of correct responses between the two tests for this question occurred in other conceptually difficult questions as well suggesting that the concepts were being gradually developed over the period of time between the two administrations of the post-test. The students did not study any calculus topics in the intervening period so the improved results could not be due to any classwork done in this time.

Discussion

Conclusions and Limitations

The results obtained in questions 1 to 3 of the post-test demonstrate that the students were able to successfully perform the standard algorithms of differential calculus. This was despite the fact that significantly less time was given to these techniques in the teaching program than is normally the case and supports the view that time spent developing concepts is not wasted in terms of improving students' skills. If the students' conceptual understanding is enhanced in this way then they do not need to spend as much time learning how to solve standard problems. Furthermore, if they attempt fewer exercises, treated in greater depth so that the underlying concepts can be more fully explored, similar or better results can be achieved without completing numerous exercises by rote.

The results of the remaining questions of the post-test also generally support the view that the methods developed in the teaching program are more successful in building conceptual understanding than the traditional first principles approach. The avoidance of difficult algebraic manipulations, complicated notation and demanding concepts such as functions and limits in the initial stages of calculus instruction is clearly beneficial. Moreover, it is also advantageous to give the calculus material a realistic context which more appropriately reflects the interests and experiences of the students.

The approach based on magnifying curves and practical tangents not only avoids these difficulties but also has advantages of its own. It provides powerful visual images which give the calculus concepts a more dynamic character and allow the students to explore these concepts in a numerical and graphical setting as well. This encourages a slower pace which affords the students more time to assimilate the new concepts and also helps to reduce the reliance on algebra and notation in the early stages (Ryan, 1991).

The principles which governed the work of Mary Barnes (1991) of group work, discovery learning and using meaningful examples presented in an appropriate context also proved to be important aspects of the teaching program. Through the interviews and diary entries it became clear that many of the students who participated in the study gained more from the experience than improved mathematical understanding. The students achieved a greater appreciation of the historical development of calculus and the ways which it can be used to solve everyday problems. They also reported positive feelings toward their work and generally found it more interesting and enjoyable than other topics they had studied.

The teaching program included a number of aspects which departed from the traditional first principles approach but it is beyond the scope of this research to identify which of these had the greater influence in improving the level of conceptual understanding attained by the students. For instance, perhaps the notion of local straightness of curves was important or it may have simply been that the use of the graphics calculators to provide visual images was in itself the key factor. No firm conclusion can be drawn at this point.

It is also not possible to infer generalisations from this study as the research is based on the study of one class without any comparison group. While it can be seen that this particular class benefited from the teaching program, this may not apply in all cases. The fact that the teacher also operated in the role of researcher may have contributed unduly to the results obtained. The positive rapport which existed between the teacher and students may have influenced in some way the willingness of the students to ensure the success of the study. Such influences are virtually impossible to identify, but they should be acknowledged in such a small study.

Implications

There are some important implications for classroom practice which follow from the results of this study. Effective calculus instruction should be characterised by placing less emphasis on algebraic processes and functions, and introducing notation in a gradual way. Limits should not be treated until after the students are more confident of their understanding of the basic calculus concepts. Instead of relying on drill and rote learning procedures, students should be given the opportunity to investigate problems and discover many of the significant calculus concepts for themselves.

Graphical and numerical approaches should be used in conjunction with algebraic methods to investigate concepts. The use of graphs when students first consider rates of change is very important and numerical calculations can be used to help establish patterns from which the general algebraic results can be derived. Time should be spent carefully developing concepts in the initial stages of calculus teaching in order to reduce some of the misunderstandings which might arise later when students consider the applications of calculus. These applications should be presented using appropriate real-world examples which show the power of calculus to solve problems.

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